Theory of unsteady flow about thin cylinders in fluids of high electrical conductivity

By LEON E. RING
Cornell University, Ithaca, N.Y.

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A theory is developed for the incompressible flow of a fluid with high electrical conductivity about thin cylinders (airfoils) in non-uniform motion. A uniform magnetic field is applied parallel to the free stream and solutions are obtained subject to the restriction of small perturbations. The effects of viscosity are included, for the most part, only through the use of the Kutta condition, where applicable, for lifting airfoils. The validity and range of applicability of the infinite-conductivity assumption are determined on the basis of an order-of-magnitude analysis; the general character of the flow is discussed at length.

The flow-field for infinite conductivity is changed from the non-magnetic case only through the new transport speed of vorticity; the forces on the airfoil are changed due to surface currents. For the case of the Alfvén speed less than the free-stream speed, the airfoil lift and pitching moment are given in integral form for general unsteady-airfoil motion and are given in closed form for harmonic ocsillations. The forces at moderate frequencies may be larger than in the corresponding non-magnetic case. The response to a unit-step change in the downwash is studied and the asymptotic form of the lift is obtained for small and large time.

For the case of the Alfvén speed greater than the free-stream speed, vorticity and current are shed from both the leading and trailing edges. Therefore the extension of the usual Kutta condition is not obvious. It is shown that if finite viscosity and/or conductivity tend to remove the trailing-edge singularity, the flow is unstable and no steady flow can be obtained.

1. Introduction

The steady magnetohydrodynamic flow of an incompressible fluid of large electrical conductivity about thin cylinders (airfoils) has been studied by Sears & Resler (1959). In the limit of infinite conductivity with a uniform magnetic field applied parallel to the free stream, the flow field is found to be irrotational and current-free and is in fact identical with the corresponding non-magnetic flow. The magnetic field inside the body is identically zero, so that the pressure on the body surface is altered due to the implied current sheet on the body surface. For large but finite conductivity, this current sheet diffuses into a thin layer. This inviscid 'magnetic-boundary-layer' has been studied by Lewellen (1959) and is found to have a thickness of the order $R_m^{-\frac{1}{2}}$, where R_m is the magnetic Reynolds number.

The results of these investigations are of sufficient general interest to suggest an analogous study of unsteady flow. In particular, since thin cylinders in asymmetric flow generally produce circulation and lift, one is led to consider the effects of unsteady circulation and vortex shedding. Thus, although ostensibly concerned with 'airfoils', the present paper is intended to cast light on phenomena that can be expected to appear in a rather general class of unsteady magneto-hydrodynamic flows around bodies. The term 'airfoil' is used since it suggests the thin cylinders being considered and since the methods of classical thin-airfoil theory are applicable to the present problem.

A free-vortex element in an incompressible and infinitely conducting fluid will travel at the Alfvén speed relative to the flow. For the aligned-fields case, this says that shed vorticity will travel parallel to the free stream, but will have a new transport speed. This suggests the use of the techniques of classical unsteady-airfoil theory, as given by Kármán & Sears (1938), with modifications for the new vortex-transport speeds and the airfoil forces due to surface currents. This is done in the present paper, so that the Sears & Resler (1959) steady solution with aligned fields is extended to include unsteady airfoil motion. Of course, this is based on the assumption, made by Sears & Resler and others, that the correct steady-flow pattern has undisturbed conditions far from the body. Other hypotheses, leading to grossly different flow patterns, have been suggested by Stewartson (1960).

2. Basic equations

The equations which govern the motion of an incompressible fluid with finite viscosity and scalar conductivity have been given by a number of authors (e.g. Resler & Sears 1958) and can be reduced to the following dimensionless form:

$$\frac{D\mathbf{q}}{Dt}+\tfrac{1}{2}\nabla(p+\alpha^2B^2) = \alpha^2\mathbf{B}\,.\nabla\mathbf{B}+\frac{1}{R_e}\nabla^2\mathbf{q}, \tag{1}$$

$$\frac{D\mathbf{B}}{Dt} = \mathbf{B}\,.\nabla\mathbf{q}+\frac{1}{R_m}\nabla^2\mathbf{B}, \tag{2}$$

$$\nabla\,.\,\mathbf{q} = 0, \tag{3}$$

$$\nabla\,.\,\mathbf{B} = 0. \tag{4}$$

The equations have been non-dimensionalized using the free-stream speed U_0^*, the applied magnetic field of induction B_0^*, and the body length L_0^*. M.K.S rationalized units are used, with an asterisk denoting dimensional quantities. The three parameters are defined by

$$R_m = \mu^*\sigma^*U_0^*L_0^*, \quad R_e = \frac{U_0^*L_0^*}{\nu^*}, \quad \alpha^2 = \frac{B_0^{*2}}{\mu^*\rho^*U_0^{*2}};$$

R_m is the magnetic Reynolds number and R_e the viscous Reynolds number. The parameter α^2 is the ratio of the magnetic pressure to the dynamic pressure in the free stream; it is also the square of the ratio of the speed of an Alfvén-wave in an infinitely conducting medium to the free-stream speed. The fluid is assumed to have uniform permeability μ^* and scalar conductivity σ^*. We will also use the vorticity $\mathbf{\Omega}$ = curl $\mathbf{q}$ and the current density $\mathbf{j}$ = curl $\mathbf{B}$.

Now consider the two-dimensional flow where small perturbations are superimposed on a uniform free-stream and a parallel magnetic field. This is represented by

$$\mathbf{q} = (1+u_x, u_y, 0), \quad u_x, u_y \ll 1;$$

$$\mathbf{B} = (1+b_x, b_y, 0), \quad b_x, b_y \ll 1.$$

For the most part, it is assumed that the flow field is described by taking zero viscosity and infinite conductivity in the linearized form of equations (1) and (2), which become

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{u}}{\partial x}+\nabla(\tfrac{1}{2}p+\alpha^2 b_x) = \alpha^2\frac{\partial \mathbf{b}}{\partial x}, \tag{5}$$

$$\frac{\partial \mathbf{b}}{\partial t}+\frac{\partial \mathbf{b}}{\partial x} = \frac{\partial \mathbf{u}}{\partial x}. \tag{6}$$

3. General character of the flow

Steady flow

Before seeking a solution, it is well to establish the general character of the flow on the basis of order-of-magnitude arguments. In particular, the validity and range of application of the infinite-conductivity assumption (to be used later) will be established. First, let us consider flow with $R_m \ll R_e$.

By comparing the first and last terms of equation (2), one may write a characteristic magnetic diffusion time as $\tau_m = R_m\delta_m^2$, where δ_m is a characteristic diffusion distance or, in the present case, the current-penetration depth normal to the body surface. In steady flow, the time available for diffusion is the time required for current to travel the length of the body. Since the current travelling with the upstream Alfvén mechanism remains in the neighbourhood of the body longer and thus allows greater time for diffusion, the characteristic convection time is taken as $\tau_c^* = L_0^*/|U_0^* - B_0^*\mu^{*-\frac{1}{2}}\rho^{*-\frac{1}{2}}|$, which in dimensionless form becomes $\tau_c = 1/|1-\alpha|$. Equating these two times, the penetration depth for a convection-diffusion balance is given by $\delta_{mc} = (R_m|1-\alpha|)^{-\frac{1}{2}}$. This agrees with the result of Sears & Resler (1959), namely that for infinite conductivity the flow field is irrotational and current-free. It also agrees with Lary (1960), who found the penetration depth to be $|k|^{-\frac{1}{2}} \equiv \{\frac{1}{2}R_m|1-\alpha^2|\}^{-\frac{1}{2}}$, since $\frac{1}{2}(1+\alpha)$ is of order unity.

The strength of the magnetic field inside the body can be estimated by noting that the magnetic flux inside the body must be equal to that which has diffused from the fluid. The flux which has diffused from the fluid can be approximated by $\delta_m \Delta B$, where ΔB is the change in the field strength across the diffusion region. The flux inside the body can be approximated by ϵB_B, where ϵ is the body thickness ratio and B_B is a measure of the field strength inside the body. Equating these, B_B is given by $B_B = \delta_m \Delta B/\epsilon$. If $\delta_m \ll \epsilon$, i.e. if the diffusion depth is much less than the body thickness, then B_B must be nearly zero since ΔB can be no larger than order unity. This gives an effective 'no-slip' condition on $\mathbf{B}$ and results in a 'magnetic-boundary-layer', analogous to the usual viscous boundary layer. If $\delta_m \gg \epsilon$, B_B can be expected to be of the order of the free-stream value and ΔB a small quantity. This agrees with the Sears & Resler result that the magnetic field is zero inside the body for infinite conductivity, and with Lary's result that the magnetic field is only slightly perturbed inside the body when $\epsilon \ll \{\frac{1}{2}R_m|1-\alpha^2|\}^{-\frac{1}{2}}$.

Unsteady flow

Now let us extend the above results to unsteady flow. Consider the pitching motion of a thin cylinder (insulator) which is described by the circular frequency ω. The unsteady characteristic time is taken as the 'period' $\tau_\omega = \omega^{-1}$. The penetration depth for a balance between diffusion and an unsteady input is found again by equating times to give $\delta_{m\omega} = \omega^{-\frac{1}{2}}R_m^{-\frac{1}{2}}$.

When $\delta_{m\omega} > \delta_{mc}$, the unsteady current component is carried away by the Alfvén mechanism before it can diffuse to the depth $\delta_{m\omega}$, so the essential balance remains between convection and diffusion. With this balance, the general flow character (except for the wake) can be expected to be the same as for steady flow.

For $\delta_{m\omega} < \delta_{mc}$, the unsteady current component is restricted to a smaller region than the steady component. Although there is a small-scale balance between diffusion and unsteady input, the large-scale balance remains between convection and diffusion. There are three cases depending upon the body thickness-ratio ϵ. (*a*) When $\delta_{m\omega} < \delta_{mc} \ll \epsilon$, both mechanisms tend toward a magnetic boundary layer. (*b*) When $\delta_{m\omega} \ll \epsilon \ll \delta_{mc}$, the unsteady current gives a magnetic boundary layer,while the steady solution gives a high degree of diffusion. This could be achieved for moderate values of ω with a high R_m and α sufficiently close to unity. This flow can be described by saying that the magnetic field is effectively 'frozen in the flow' during an oscillation period but can diffuse a large distance during the time that the current stays in the neighbourhood of the body. (It is also 'frozen out of the flow' or 'in the body' during an oscillation period.) (*c*) When $\epsilon \ll \delta_{m\omega} < \delta_{mc}$, both mechanisms allow a high degree of diffusion.

For the slender bodies being considered, the boundary conditions at the body-fluid interface can be satisfied by discontinuities across a singularity sheet, so the solutions are linear and superposition of solutions is possible (for thickness, camber, unsteadiness, etc.). That this is true for the limiting cases discussed above follows directly from the work of Sears & Resler (1959) and Lary (1960) except for the special case $\delta_{m\omega} \ll \epsilon \ll \delta_{mc}$. That it is also true in this case can be seen by noting that the steady solution has only small perturbations everywhere. Therefore the unsteady solution, within the linear theory, sees only an undisturbed flow. The boundary conditions for the solution of the unsteady flow field can thus be satisfied by a singularity sheet. The required unsteady current-sheet strength corresponds to the jump from the unsteady flow to the steady magnetic field inside the body, since the field strength inside the body cannot change during an oscillation period.

From the above, it is now possible to answer the question of the applicability of the infinite-conductivity theory. For the calculation of thickness effects, in general it is required that the magnetic field be approximately zero inside the body; this requires that $\epsilon \gg (R_m|1-\alpha|)^{-\frac{1}{2}}$. However, the unsteady infinite-conductivity lifting theory only requires that the unsteady current be confined to a thin layer; thus infinite-conductivity theory is valid when *either*

$$1 \ll (R_m|1-\alpha|)^{\frac{1}{2}} \quad \text{or} \quad 1 \ll (\omega R_m)^{\frac{1}{2}}.$$

A case of particular interest occurs when $1 \ll (\omega R_m)^{\frac{1}{2}}$ and $1 \gg (R_m|1-\alpha|)^{\frac{1}{2}}$. The solution for this flow is given by the supposition of a steady contribution, calculated on the basis of Lary's (1960) theory, and an unsteady contribution calculated on the basis of the infinite-conductivity theory of the present paper.

4. Conservation laws and Kutta condition

Conservation of vorticity

Consider a body at rest which starts to move at time $t = 0$. At any finite time t, all current and vorticity must be in a finite region of the plane which can be enclosed by a curve C, where C is convected with the local flow. Since $\mathbf{\Omega}$ and $\mathbf{j}$ are assumed to be zero along C, Kelvin's theorem can be applied to the curve C and one can immediately write that

$$\iint \mathbf{\Omega}\, dx\, dy = 0, \tag{7}$$

where the integral is taken over all the vorticity, that is, the net vorticity in the system remains zero.

Conservation of current

A similar constraint will now be obtained for the current. Consider the contour C to be a circle of radius r about the body, and integrate equation (2) around C. This gives

$$\frac{d}{dt}\int_C \mathbf{B}.d\mathbf{l} = \int_C \left(-\mathbf{q}.\nabla\mathbf{B}+\mathbf{B}.\nabla\mathbf{q}+\frac{1}{R_m}\nabla^2\mathbf{B}\right).d\mathbf{l}. \tag{8}$$

Since there is no current on C, $\nabla^2\mathbf{B} = 0$ on C. The net vorticity was shown to be zero, so $\mathbf{q} = O(r^{-2})$ on C. Therefore the right-hand side of (8) is $O(r^{-1})$, which can be neglected for a sufficiently large contour. Then, applying Stokes's theorem and integrating over time, we have that

$$\iint \mathbf{j}\, dx\, dy = \text{const.} = 0, \tag{9}$$

where the integral is taken over all current, and the integration constant is evaluated at a time before the start of the body motion. This says that the net current in the system remains zero.†

† One must be a bit careful about what is meant by a two-dimensional problem, since all currents must close somehow. If there is a net current flowing, there is some question if the closing of the current (in a large wire loop, for instance) can be neglected. In the present work, the net current is zero. Therefore the current elements can close through conducting side walls, which would give an effect analogous to tip effects of classical-airfoil theory.

Kutta condition

In classical two-dimensional thin-airfoil theory, the undetermined cyclic constant (circulation) is specified, to make the solution unique, by using the Kutta condition. The criterion for small, but finite, viscosity has been discussed by a number of authors, including a recent comprehensive discussion by Sears (1956). It is because of the fact that vorticity convects with the flow (and diffuses relative thereto) that the boundary-layer vorticity leaves the airfoil at the trailing edge; therefore the Kutta condition must be applied at this edge.

For a magnetic boundary layer with $\alpha^2 < 1$, all vorticity and current must leave the airfoil from the trailing edge. Thus the Kutta condition must still be applied at the trailing edge. For $\alpha^2 > 1$, the extension is not clear. It can be shown that if the angle of attack is suddenly changed, there is an immediate flux of vorticity and current from both the leading and trailing edges. Therefore it seems that, at least in the unsteady case, the criterion should include the leading as well as the trailing edge. Since we lack such a criterion, which must come from more detailed boundary-layer considerations, this case will be treated without applying a specific condition.

5. General unsteady airfoil motion

Fluid model

Consider the unsteady two-dimensional flow about a thin cylinder (airfoil) for arbitrary values of the parameter α^2. The airfoil is taken to have zero thickness and to lie near the x-axis in the range $[0, 1]$. If the fluid has sufficiently high electrical conductivity and low viscosity, all current and vorticity are confined to thin wakes and the wakes can be represented as singularity sheets.

The fluid model used for the solution of the flow field is shown in figure 1, and consists of a current and vortex distribution in a downstream wake, over the airfoil, and when applicable, in an upstream wake. The current and vorticity in the wakes can be related by the solution of equation (5) and (6) with the condition that the wakes are force-free. This gives the well known Alfvén-wave relations,

$$\left.\begin{aligned} \epsilon_1^{(1)}(x,t) &= \alpha\epsilon_1^{(2)}(x,t) = f_1\{t-x/(1-\alpha)\}, \\ \epsilon_2^{(1)}(x,t) &= -\alpha\epsilon_2^{(2)}(x,t) = f_2\{t-x/(1+\alpha)\}, \end{aligned}\right\} \tag{10}$$

where $\epsilon^{(1)}$ is wake vorticity, $\epsilon^{(2)}$ is wake current, and the subscripts 1 and 2 refer to upstream and downstream waves respectively. These have velocities and direction of travel as shown in figure 1. For any finite time, the wakes will be of finite length and all current and vorticity will lie in the interval $-R < x < R$.

Potential relations

The flow outside the wakes can be given by a perturbation magnetic potential $\phi^{(2)}$ and a perturbation velocity potential $\phi^{(1)}$. A relation between the potentials is obtained by substitution into equation (6). Integration of the x-component with respect to x yields the result

$$\frac{\partial\phi^{(2)}}{\partial t} + \frac{\partial\phi^{(2)}}{\partial x} = \frac{\partial\phi^{(1)}}{\partial x},$$

where the integration constant is found to be zero, by evaluation at $x = \infty$. $\phi^{(2)}$ is now given explicitly in terms of $\phi^{(1)}$ by integration of the above result, following a fluid particle. This yields the integral

$$\phi^{(2)}(x,y,t) = \int_{-\infty}^{t} \frac{\partial\phi^{(1)}(x-t+\xi, y, \xi)}{\partial x} d\xi \tag{11a}$$

$$= \phi^{(1)}(x,y,t) - \int_{-\infty}^{x} \frac{\partial\phi^{(1)}(\xi, y, t+\xi-x)}{\partial t} d\xi. \tag{11b}$$

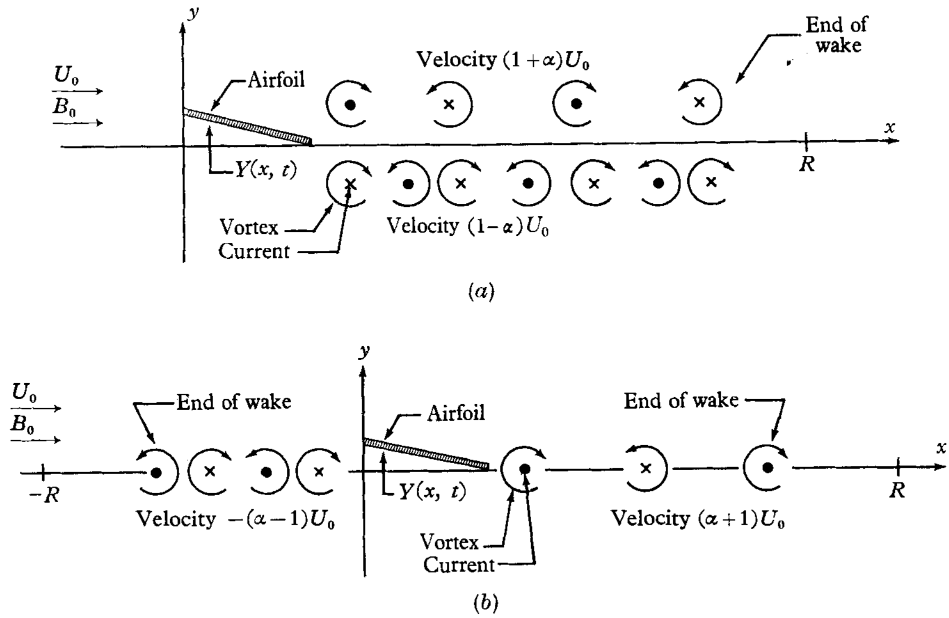


FIGURE 1. Fluid model for a lifting airfoil which has been in motion a finite time, with (*a*) $\alpha^2 < 1$, (*b*) $\alpha^2 > 1$. Wake elements have been displaced from x-axis for clarity.

The potentials are taken to be identically zero at $t = -\infty$. The magnetic and velocity potentials must satisfy equation (11) at every point in the flow field. Therefore the discontinuities must also satisfy this relation, and the current distribution $\gamma^{(2)}$ can be expressed in terms of the vortex distribution $\gamma^{(1)}$ in the same form as equation (11) (with no y dependence). The vorticity and current in the wake are expressed as

$$\gamma^{(1),(2)}(x,t) \equiv \epsilon^{(1),(2)}(x,t) \equiv \epsilon_1^{(1),(2)}(x,t) + \epsilon_2^{(1),(2)}(x,t), \tag{12}$$

where the Alfvén-wave relations are given in (10).

The circulation and total surface current are related to the distributions by

$$\Gamma^{(1),(2)}(t) = \int_0^1 \gamma^{(1),(2)}(x,t)\, dx, \tag{13}$$

where the circulation is positive in the counter-clockwise sense. The total surface current can be expressed in terms of the vortex distribution by using the current equivalent of equation (11) in equation (13) and the functional form of the wake from equation (10). Thus we derive the result

$$\Gamma^{(2)}(t) = \int_0^1 \gamma^{(1)}(\xi, t+\xi-1)\, d\xi + \frac{1-\alpha}{\alpha}\int_{t-1}^{t} \epsilon_1^{(1)}(0^-, \eta)\, d\eta. \tag{14}$$

The first term on the right-hand side is the vortex distribution integrated in a time-lagging manner; the second is proportional to the vorticity shed from the leading edge in the previous unit of time. These indicate the general time-lagging character of the magnetic field to the velocity field.

The analogous Wagner integral equations

In classical unsteady-airfoil theory, the airfoil motion is related to the wake vorticity through the so-called Wagner integral equation. If we follow the techniques of the classical theory (e.g. von Kármán & Sears 1938), we obtain the generalization:

$$\Gamma_0^{(1)}(t) + \Gamma_k^{(1)}(t) = -\int_{\text{wake}} \epsilon^{(1)}(\xi, t)\left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}} d\xi. \tag{15}$$

Here the quasi-steady circulation is given by

$$\Gamma_0^{(1)}(t) = 2\int_0^1 \frac{DY(\xi,t)}{Dt}\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi, \tag{16}$$

and the airfoil surface is given by $Y(x,t)$. $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x}$ is the linearized convective derivative. $\Gamma_k^{(1)}$ represents an arbitrary circulation; if the Kutta condition is applied at the trailing edge, then $\Gamma_k^{(1)}$ is identically zero. The integration over the wake includes the forward as well as the rearward wake. It should be noted that the conservation of vorticity, equation (7), is used in the derivation of (15).

The total surface current is expressed in terms of the vortex distribution by (14). Using this, an analysis similar to that leading to equation (15) yields the result

$$\Gamma_0^{(2)}(t) + \Gamma_k^{(2)}(t) = -\int_{\text{wake}} d\xi \left\{\epsilon^{(2)}(\xi,t) - \frac{1}{\pi}\left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}} \int_0^1 \epsilon^{(1)}(\xi, t+x-1)\left(\frac{1-x}{x}\right)^{\frac{1}{2}} \frac{dx}{x-\xi}\right\}$$
$$- \frac{1-\alpha}{\alpha}\int_{t-1}^{t} \epsilon_1^{(1)}(0^-, \eta)\, d\eta, \tag{17}$$

where the quasi-steady surface current is given by

$$\Gamma_0^{(2)}(t) = -\frac{2}{\pi}\int_0^1 \left(\frac{1-x}{x}\right)^{\frac{1}{2}} dx\, \mathscr{P}\int_0^1 \frac{DY(\xi, t+x-1)}{Dt}\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} \frac{d\xi}{x-\xi}, \tag{18}$$

and the surface current corresponding to the arbitrary circulation is

$$\Gamma_k^{(2)}(t) = \frac{1}{\pi}\int_0^1 \Gamma_k^{(1)}(t+x-1)\frac{dx}{\{x(1-x)\}^{\frac{1}{2}}}. \tag{19}$$

Equation (17) is the analogue of the Wagner integral equation for current.

The problem is now reduced to the solution of the integral equations (15) and (17), with the quasi-steady terms defined by (16) and (18), with the functional form of the wake given in (10) and given some law to specify the circulation $\Gamma_k^{(1)}$ (e.g. the Kutta condition).

The lift

Consider the lift due to the vortex and current distribution over the airfoil and in the wake. Since the wake is force-free, it can introduce no net force. Divide the lift into that due to the vorticity and that due to the Lorentz force. The lift due to the Lorentz force is

$$\Delta \mathscr{L}^* = B^* \iint j^* dx^* dy^*, \tag{20}$$

where the integral includes all current. This integration just gives the total current in the system, and, by equation (9), this is identically zero. Therefore the lift on the airfoil is equal to the force on the total vorticity in the system. The force on the vorticity can be obtained by calculating the impulse of the vortex system, as given by Kármán & Sears (1938); this gives

$$C_L(t) = \frac{\mathscr{L}^*}{\frac{1}{2}\rho^* U_0^{*2} L_0^*} = 2\frac{d}{dt}\int_{-R}^{R} (x-\tfrac{1}{2})\,\gamma^{(1)}(x,t)\,dx. \tag{21}$$

The lift is treated following the method of Kármán & Sears with modification for the change in vortex transport speeds, used by Ring (1960). The result, including arbitrary circulation and a possible forward wake, is

$$C_L(t) = A_L - 2\Gamma_0^{(1)}(t) - 2\Gamma_k^{(1)}(t) - \int_{\text{wake}} \epsilon^{(1)}(\xi,t)\left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}}\frac{d\xi}{\xi}$$
$$- 2\alpha^2 \int_{\text{wake}} \epsilon^{(2)}(\xi,t)\left\{1-\frac{1}{2\xi}\right\}\left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}} d\xi, \tag{22}$$

where A_L is the lift associated with apparent mass and is given by

$$A_L \equiv 2\frac{d}{dt}\int_0^1 (x-\tfrac{1}{2})\,\gamma_0^{(1)}(x,t)\,dx. \tag{23}$$

Here $\gamma_0^{(1)}$ is the quasi-steady vortex distribution over the airfoil,

$$\gamma_0^{(1)}(x,t) = -\frac{2}{\pi}\left(\frac{1-x}{x}\right)^{\frac{1}{2}} \mathscr{P}\!\int_0^1 \frac{DY(\xi,t)}{Dt}\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}}\frac{d\xi}{x-\xi}. \tag{24}$$

From equation (22), the lift can be compared to that of classical unsteady-airfoil theory. The apparent mass and the quasi-steady lift (the second term) are identical with classical theory. The third term represents the effect of an arbitrary circulation and is identically zero for a trailing-edge Kutta condition. The effect of vorticity in the wake (the fourth term) is formally the same, but is changed due to the new transport speed of vortices. The last term is new and might be called the magnetic term. The kernel of the magnetic term is seen to approach unity far from the airfoil, unlike the kernel of the wake-vorticity term which approaches ξ^{-1}. This means that the magnetic term takes into account all current in the wake and gives a contribution even after the flow has become steady near the airfoil. In fact, it can easily be shown that the magnetic term includes the effects of surface currents in the steady solution and gives rise to the $(1-\alpha^2)$ factor of Sears & Resler (1959).

The pitching moment

As in the lift calculation, we divide the pitching moment into that due to the vorticity and that due to the currents. The pitching moment coefficient about the airfoil midchord ($x = \frac{1}{2}$), measured positive in the clockwise sense, due to the Lorentz force can be written as

$$\Delta C_m = \frac{\Delta \mathscr{M}^*}{\frac{1}{2}\rho^* U_0^{*2} L_0^{*2}} = -2\alpha^2 \int_{-R}^{R} (x-\tfrac{1}{2})\,\gamma^{(2)}(x,t)\,dx. \tag{25}$$

The current can be expressed in terms of the vorticity in a manner similar to the potential relation (11 *a*), namely

$$\Delta C_m = -2\alpha^2 \int_{-R}^{R} (x-\tfrac{1}{2})\,dx \int_{-\infty}^{t} \frac{\partial \gamma^{(1)}}{\partial x}(\xi + x - t, \xi)\,d\xi. \tag{26}$$

We differentiate this with respect to time and then integrate by parts with respect to x; then, by use of (11 *a*), this becomes

$$\frac{d\Delta C_m}{dt} = -2\alpha^2 \bigg\{ (R-\tfrac{1}{2})\,[\gamma^{(1)}(R,t) - \gamma^{(2)}(R,t)] + (R+\tfrac{1}{2})\,[\gamma^{(1)}(-R,t) - \gamma^{(2)}(-R,t)] - \int_{-R}^{R} [\gamma^{(1)}(x,t) - \gamma^{(2)}(x,t)]\,dx \bigg\}. \tag{27}$$

The first two terms on the right-hand side are zero, since they represent vorticity and current beyond the ends of the wake; the last term is the net vorticity minus the net current in the system and therefore is zero. By integration over time, the incremental pitching moment is found to be given by

$$\Delta C_m = \text{constant} = 0, \tag{28}$$

where the constant is evaluated at a time before the start of the airfoil motion. Therefore the surprising result is obtained that the pitching moment due to the total current distribution is zero. The lift due to the current is zero essentially as a consequence of the conservation law for current; however, it is not obvious why the pitching moment should have been zero.

The pitching moment due to the vorticity can be found by using impulse techniques. Again following the method of Kármán & Sears as modified by Ring, the pitching moment can be expressed as

$$C_m(t) = A_m + 2\int_0^1 (x-\tfrac{1}{2})\,\gamma_0^{(1)}(x,t)\,dx - \frac{1}{4}\int_{\text{wake}} \epsilon^{(1)}(\xi,t) \left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}} \frac{d\xi}{\xi} + 2\alpha^2 \int_{\text{wake}} \epsilon^{(2)}(\xi,t)\,\{\xi - 1 + \tfrac{1}{8}\xi^{-1}\} \left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}} d\xi, \tag{29}$$

where A_m is the pitching moment related to apparent-mass effect and is defined by

$$A_m \equiv -\frac{d}{dt}\int_0^1 [(x-\tfrac{1}{2})^2 - \tfrac{1}{8}]\,\gamma_0^{(1)}(x,t)\,dx. \tag{30}$$

The first three terms are recognized as those from classical unsteady-airfoil theory. The wake-vorticity term is formally the same but is changed because of the new transport speeds of vorticity. The last term is new and might be called the magnetic term.

As the duration of the airfoil motion becomes large, and therefore the current is far from the airfoil, it appears that the magnetic term diverges, since the integrand approaches $\xi\epsilon^{(2)}(\xi,t)$. However, that this is not actually the case can be seen by subtracting the magnetic moments due to surface current on the airfoil (expressed in terms of the wake from equations (25) and (28)). In fact, the magnetic term again gives rise to the $(1-\alpha^2)$ factor of Sears & Resler (1959).

6. Harmonic oscillations, $\alpha^2 < 1$

Calculation of wake distributions

Consider an oscillating airfoil with the upwash on the airfoil given by

$$\frac{DY(x,t)}{Dt} = g(x)\,e^{i\omega t}, \tag{31}$$

where $g(x)$ is an arbitrary function. The frequency ω is taken such that $\mathscr{R}i\omega > 0$; this corresponds to an airfoil oscillating with increasing amplitude. Clearly $\mathscr{R}i\omega$ can be put equal to zero in the final results.

Following §4, the Kutta condition is applied at the airfoil trailing edge, since $\alpha^2 < 1$. This means that $\Gamma_k^{(1)}$ must be identically zero.

The quasi-steady circulation and surface current can be expressed as

$$\Gamma_0^{(1)} = G^{(1)}e^{i\omega t}, \quad \Gamma_0^{(2)} = G^{(2)}e^{i\omega(t-\frac{1}{2})}, \tag{32}$$

where $G^{(1)}$ and $G^{(2)}$ are complex in general and are given by using equation (31) in equations (16) and (18). Thus we obtain the relations

$$G^{(1)} = 2\int_0^1 g(\xi)\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi, \tag{33}$$

$$G^{(2)} = -\frac{2}{\pi}\int_0^1 \left(\frac{1-x}{x}\right)^{\frac{1}{2}} e^{i\omega(x-\frac{1}{2})}\,dx\,\mathscr{P}\!\int_0^1 g(\xi)\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}}\frac{d\xi}{x-\xi}. \tag{34}$$

From the functional form of the wake given in (10), the wake distributions can be written as

$$\left.\begin{aligned}\epsilon_1^{(1)}(x,t) &= \alpha\epsilon_1^{(2)}(x,t) = g_1\exp\{i\omega[t-(x-\tfrac{1}{2})/(1-\alpha)]\},\\ \epsilon_2^{(1)}(x,t) &= -\alpha\epsilon_2^{(2)}(x,t) = g_2\exp\{i\omega[t-(x-\tfrac{1}{2})/(1+\alpha)]\}.\end{aligned}\right\} \tag{35}$$

where g_1 and g_2 are complex constants to be determined. Using equations (32) and (35), equation (15) can be reduced to

$$-2G^{(1)} = g_1[K_0^{(1)} + K_1^{(1)}] + g_2[K_0^{(2)} + K_1^{(2)}], \tag{36}$$

where the integrals have been identified as Bessel functions; $K_n(z)$ is the nth-order modified Bessel function of the second kind. The superscripts are a shorthand notation to indicate the arguments, as shown below:

$$K_0^{(1)} \equiv \frac{1-\alpha}{|1-\alpha|} K_0\left(\frac{i\omega}{2|1-\alpha|}\right), \quad K_1^{(1)} \equiv K_1\left(\frac{i\omega}{2|1-\alpha|}\right), \quad K_n^{(2)} \equiv K_n\left(\frac{i\omega}{2(1+\alpha)}\right). \tag{37}$$

The factor $(1-\alpha)/|1-\alpha|$, which is unity in the present case, is included to make the definitions useful for $\alpha^2 > 1$. Similarly, equation (17) can be written as

$$-2G^{(2)} = g_1 D(\omega,\alpha) + g_2 D(\omega, -\alpha), \tag{38}$$

where D is defined by

$$D(\omega,\alpha) \equiv \frac{2}{\alpha}\exp\left(\tfrac{1}{2}i\omega\right)\int_1^\infty \exp\left\{\frac{i\omega(\xi-\frac{1}{2})}{(1-\alpha)}\right\}d\xi$$
$$-\frac{2}{\pi}\int_1^\infty \left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}} \exp\left\{-\frac{i\omega(\xi-\frac{1}{2})}{1-\alpha}\right\}d\xi \int_0^1 \left(\frac{1-x}{x}\right)^{\frac{1}{2}} \exp\{i\omega(x-\tfrac{1}{2})\}\frac{dx}{x-\xi}. \tag{39}$$

The function D has been evaluated in Ring (1960) and is

$$D(\omega,\alpha) = [J_0 - iJ_1]K_0^{(1)} + \alpha^{-1}[J_0 K_1^{(1)} + iJ_1 K_0^{(1)}], \tag{40}$$

where $J_n(z)$ is the nth-order Bessel function of the first kind. The implied arguments of J_0 and J_1 are $\frac{1}{2}\omega$. The simultaneous solution of equation (36) and (38) gives g_1 as

$$g_1 = -2G^{(1)} \frac{D(\omega,-\alpha) - G[K_0^{(2)} + K_1^{(2)}]}{D(\omega,-\alpha)[K_0^{(1)} + K_1^{(1)}] - D(\omega,\alpha)[K_0^{(2)} + K_1^{(2)}]}, \tag{41}$$

where

$$G \equiv G^{(2)}/G^{(1)}. \tag{42}$$

The constant g_2 is given by taking $\alpha \to -\alpha$ in the expression for g_1. This essentially represents the solution of the flow, since the lift, pressure distribution, etc., can easily be expressed in terms of $G^{(1),(2)}$ and $g_{1,2}$.

Force calculations

The lift has been given in equation (22); with the use of (35) and (41), this can be reduced to

$$C_L(t) = A_L - 2\Gamma_0^{(1)}(t)\,T(\tfrac{1}{2}i\omega,\alpha), \tag{43}$$

where

$$T(\tfrac{1}{2}i\omega,\alpha) \equiv \frac{(1-\alpha)K_1^{(1)}\{D(\omega,-\alpha) - G[K_0^{(2)} + K_1^{(2)}]\} - (1+\alpha)K_1^{(2)}\{D(\omega,\alpha) - G[K_0^{(1)} + K_1^{(1)}]\}}{D(\omega,-\alpha)[K_0^{(1)} + K_1^{(1)}] - D(\omega,\alpha)[K_0^{(2)} + K_1^{(2)}]}. \tag{44}$$

$T(\frac{1}{2}i\omega,\alpha)$ is the so-called Theodorsen function of classical unsteady-airfoil theory, modified to include the magnetic effects. (It should be noted that ω is the reduced frequency based on the full chord, as opposed to the usual practice of basing it on the semi-chord.) The most important change in the Theodorsen function is the appearance of G. Thus, while the classical Theodorsen function depends only on frequency, the modified Theodorsen function depends upon the mode of oscillation (and on α) as well.

In a manner similar to the life calculation, the pitching-moment is found to be

$$C_m(t) = A_m + \int_0^1 (2x-1)\,\gamma_0^{(1)}(x,t)\,dx + \tfrac{1}{2}\Gamma_0^{(1)}(t)\,S(\tfrac{1}{2}i\omega,\alpha) \tag{45}$$

where

$$S(\tfrac{1}{2}i\omega,\alpha) \equiv \frac{\begin{array}{c}(1-\alpha)\,[K_0^{(1)}-(4\alpha/i\omega)\,K_1^{(1)}]\,[D(\omega,-\alpha)-G\{K_0^{(2)}+K_1^{(2)}\}] \\ -(1+\alpha)\,[K_0^{(2)}+(4\alpha/i\omega)\,K_1^{(2)}]\,[D(\omega,\alpha)-G\{K_0^{(1)}+K_1^{(1)}\}]\end{array}}{D(\omega,-\alpha)\,[K_0^{(1)}+K_1^{(1)}]-D(\omega,\alpha)\,[K_0^{(2)}+K_1^{(2)}]}. \tag{46}$$

The function $S(\frac{1}{2}i\omega,\alpha)$ includes the wake and all magnetic effects. The appearance of G again shows that the function is not universal but depends upon the mode of oscillation considered.

Three G's are of particular interest, namely,

$$G_{\mathrm{I}} = J_0 - iJ_1, \quad G_{\mathrm{II}} = 4J_1/\omega, \quad G_{\mathrm{III}} = \tfrac{1}{2}[J_0 - iJ_1] + 2J_1/\omega. \tag{47}$$

Case I corresponds to vertical airfoil oscillations, case II to a downwash linear about the mid-chord, case III to a downwash distribution linear about the quarter-chord. The first two are of interest because of their mathematical simplicity, the last because of a special high-frequency property which will become apparent later.

Low-frequency limit

The low-frequency expansions of T and S are obtained by expanding G in a power series and by using the small-argument Bessel-function expansions. After straightforward but lengthy algebraic manipulation, the force coefficients are found to be

$$\begin{aligned} C_L(t) = {} & (1+\alpha^2)\,A_L - 2\Gamma_0^{(1)}(t)\,\{(1-\alpha^2) + \tfrac{1}{2}(1+\alpha^2)\,i\omega\ln\tfrac{1}{4}\omega + \tfrac{1}{2}[(1+\alpha^2) \\ & \times\{\gamma + \tfrac{1}{2}i\pi - \tfrac{1}{2}\ln(1-\alpha^2)\} + \alpha\ln(1+\alpha)/(1-\alpha)]\,i\omega + O(\tfrac{1}{16}\omega^2\ln^2\tfrac{1}{4}\omega)\}, \end{aligned} \tag{48}$$

$$C_m(t) = A_m + (1-\alpha^2)\int_0^1 (2x-1)\,\gamma_0^{(1)}(x,t)\,dx - \tfrac{1}{2}\Gamma_0^{(1)}(t)\,\{\tfrac{1}{2}(1+\alpha^2)\,i\omega\ln\tfrac{1}{4}\omega + O(\omega)\}. \tag{49}$$

The apparent-mass term is included in the pitching-moment although it is of order ω and some terms of order ω have been neglected.

The first thing to note from these expressions is that as $\omega \to 0$ the force coefficients become $(1-\alpha^2)$ times the values for non-magnetic flow in agreement with the Sears–Resler result. The coefficient of the $(\omega\ln\omega)$ term is increased by the factor $(1+\alpha^2)$; the coefficient of ω is also increased because of magnetic effects. Thus, although the steady forces are reduced by a factor $(1-\alpha^2)$, there is a general tendency for the unsteady contribution to be increased by the magnetic effects.

The limit $\alpha \to 0$

Taking the limit $\alpha \to 0$ (i.e. the magnetic field approaching zero), it is seen that

$$T(i\omega,\alpha) = \frac{K_1(i\omega)}{K_0(i\omega)+K_1(i\omega)} + O(\alpha^2), \tag{50}$$

and

$$S(i\omega,\alpha) = \frac{K_0(i\omega)}{K_0(i\omega)+K_1(i\omega)} + O(\alpha^2). \tag{51}$$

These limiting forms are identical with classical theory. The coefficient of the α^2 terms can be obtained; however, it turns out that the frequency dependence is of such a complicated nature as to yield little information.

High-frequency limit

As $\omega \rightarrow \infty$, it is convenient to represent G by an asymptotic expansion of the form

$$G \sim \left(\frac{4i}{\pi\omega}\right)^{\frac{1}{2}} e^{-\frac{1}{2}i\omega}\left\{\hat{G}+\frac{1}{i\omega}[-i\,e^{i\omega}\hat{\hat{G}}^{+}+\hat{\hat{G}}^{-}]+\ldots\right\}. \tag{52}$$

Using this expansion of G, the asymptotic expansion of T is

$$T(\tfrac{1}{2}i\omega,\alpha) \sim \tfrac{1}{2} \pm \tfrac{1}{2}i\alpha^2(1-2\hat{G})\,e^{-i\omega}+(4i\omega)^{-1}\{1+2\alpha^2-4\alpha^2\hat{\hat{G}}^{+} \pm i\alpha^2[1-5\hat{G}-4\hat{\hat{G}}^{-}]\,e^{-i\omega}+\alpha^4(1-2\hat{G})\,e^{-2i\omega}\}+\ldots. \tag{53}$$

This expansion is valid for $-\pi \leqslant \arg\omega \leqslant 0$; the top sign is to be used for $\arg i\omega > 0$ and the bottom sign for $\arg i\omega < 0$. The expansion of S is similar. For $\alpha = 0$, T is asymptotic to the constant $\frac{1}{2}$, as is known from classical theory. However, the general case represents a decidedly different behaviour. If viewed in the complex plane, as ω becomes large (and real), T describes a circle of radius $\frac{1}{2}\alpha^2(1-2\hat{G})$ about the point $\frac{1}{2}$. It approaches this circle in an oscillating manner, as seen from the second term in the expansion. Thus, while the steady-flow theories of Lary (1960) and of Sears & Resler (1959) predict small forces in the neighbourhood of $\alpha = 1$, the present theory predicts large forces due to unsteady effects.

Case III (equation (47)) has the special property that $(1-2\hat{G}) = 0$; in fact, it is the only linear downwash distribution to have this property. The reason why this is true will be pointed out in the discussion of the Wagner problem (§7).

The limit $\alpha \rightarrow 1$

There is some question as to the validity of the solution for $\alpha = 1$ (e.g., the Kutta condition). However, the limiting solution is of interest since it will shed some light on the solution for α near but not equal to unity. For $\alpha \rightarrow 1^-$, we obtain

$$T(\tfrac{1}{2}i\omega,1^-) = K_1^{(2)}\frac{J_0-G}{J_0K_1^{(2)}+iJ_1K_0^{(2)}}, \tag{54}$$

$$S(\tfrac{1}{2}i\omega,1^-) = T(\tfrac{1}{2}i\omega,1^-)\left[\frac{K_0^{(2)}}{K_1^{(2)}}+\frac{4}{i\omega}\right]. \tag{55}$$

The behaviour of these functions for large frequencies is similar to the general case, except that the radius of the asymptotic circle has been maximized. It is not valid to take the limit $\omega \rightarrow 0$ in these functions, since for α near unity we required that $1 \ll (\omega R_m)^{\frac{1}{2}}$ (§3). However, for large R_m, ω can be quite small. Thus we can consider small frequencies, keeping in mind the physical requirement that ω must be positive. Using the low-frequency expansions of equations (54) and (55), the force coefficients are

$$C_L(t) = 2A_L+O(\omega^2), \quad Cm(t) = O(\omega). \tag{56}$$

Thus for low frequencies and α near unity, the forces on the airfoil are small.

The circulation for $\alpha \to 1^-$ and small ω is

$$\Gamma^{(1)}(t) = -\tfrac{1}{4} i\omega \int_0^1 (2x-1)\,\gamma_0^{(1)}(x,t)\,dx\,[1+O(\omega)], \tag{57}$$

so that the circulation is small. For zero circulation, the flow pattern must correspond to classical hydrodynamic flow with a relaxed Kutta condition. Therefore the flow pattern approached by taking $\omega \to 0$ with $\alpha = 1^-$ has zero lift with zero circulation, as opposed to the limit in the Sears–Resler theory (and in the present theory), $\alpha \to 1$, with $\omega = 0$, which gives zero lift with $\Gamma^{(1)} \not\equiv 0$. Following § 3, both limits may be valid for $R_m = \infty$, with neither valid for $R_m < \infty$. However the former limit can be approached to any desired degree of accuracy by taking R_m arbitrarily large but finite.

7. The Wagner problem, $\alpha^2 < 1$

Formulation of problem

The classical problem of the response to an instantaneous change in the downwash distribution on a thin airfoil is referred to as the 'Wagner problem'. The downwash for this case may be represented by

$$\frac{DY(x,t)}{Dt} = g(x)\mathrm{H}(t), \tag{58}$$

where $\mathrm{H}(t)$ is the unit-step function defined by

$$\mathrm{H}(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

The downwash of equation (58) can be obtained by Fourier superposition from equation (31). Similarly, all quantities from the harmonic-oscillation solution may be superimposed to give the corresponding quantities caused by the instantaneous change of downwash.

The quasi-steady circulation is obtained from equation (32) as

$$\Gamma_0^{(1)}(t) = G^{(1)}\mathrm{H}(t) \tag{59}$$

The lift, obtained from equation (43) is

$$C_L(t) = \delta(t)\int_0^1 (2x-1)\,\gamma_0^{(1)}(x,0+)\,dx - 2\Gamma_0^{(1)}(t)\,W(t,\alpha), \tag{60}$$

where
$$W(t,\alpha) \equiv \frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} T(\tfrac{1}{2}s,\alpha)\frac{e^{st}}{s}\,ds, \tag{61}$$

and $\delta(t)$ is the Dirac delta function. The integral has been expressed as a Laplace-transform by taking $s = i\omega$ and x_0 to lie to the right of any singularities. The apparent mass has given an impulse lift at $t = 0$; the lift at later times is all due to the modified Wagner function $W(t,\alpha)$. Let us confine our attention to this function.

$W(t, \alpha)$ for small times

The standard method of obtaining the behaviour of W for small times is to obtain the asymptotic expansion of T for $s \to \infty$, and then relate this to the asymptotic expansion of W for $t \to 0$. The expansion of $T(\frac{1}{2}s, \alpha)$ for $s \to \infty$ is given in equation (53) and contains exponential terms. It has been shown by Ring (1960) that the e^{-s} terms give a contribution like $\exp(-1/t)$ for small times. Therefore, using known results (e.g. Doetsch 1950), the expansion of W for small times is given, from equations (61) and (53), as

$$W(t, \alpha) = \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{2}\alpha^2 - \alpha^2 \hat{\hat{G}}^{+})\, t + O(t^2). \tag{62}$$

The first term in W remains equal to $\frac{1}{2}$, unaffected by the magnetic field. The first magnetic effects are proportional to time and, since $\hat{\hat{G}}^{+}$ has appeared, depend upon the downwash distribution considered.

The result given in equation (62) will be valid even for small R_m, provided the time is taken so small that the current is still restricted to a thin layer. Following §3, this will be true for $t \leqslant 10^{-2} R_m$, where the limit on the current-penetration depth has arbitrarily been taken as $\delta_m = 10^{-1}$.

The logarithmic singularity

From the asymptotic expansion (53), it is seen that convergence of the integral in equation (61) is obtained for all time except at $t = 1$ where e^{st} balances the factor e^{-s} and gives rise to a logarithmic singularity. By extracting the coefficient of the singularity, W is found to be given by

$$W(t, \alpha) = -\tfrac{1}{2}\pi^{-1}\alpha^2(1 - 2\hat{G}) \ln |t - 1| + O(1). \tag{63}$$

This represents a marked change from the classical Wagner function, which has no singularities. The singularity is seen to arise from the high-frequency oscillating behaviour of T; having connected these phenomena, it is now possible to examine their origin.

The logarithmic singularity in W appears at time $t = 1$. At $t = 1$, the fluid at the trailing edge is that which was at the leading edge when the downwash was changed. Thus we are led to the examination of the flow near the leading edge at high frequencies. A calculation of the vortex distribution yields

$$\gamma^{(1)}(x, t) = -\pi^{-1}\Gamma_0^{(1)}(t)\{(1 - x)/x\}^{\frac{1}{2}}\{(1 - 2\hat{G}) + O(1/\omega) + O(x)\}. \tag{64}$$

Thus the coefficient of the high-frequency oscillations of $T(\frac{1}{2}i\omega, \alpha)$, the coefficient of the logarithmic singularity of $W(t, \alpha)$, and the coefficient of the leading-edge vortex singularity at high frequencies all contain the factor $(1 - 2\hat{G})$. Thus it appears that the leading-edge vortex singularity interacts with the magnetic field to produce the observed behaviour of T and W.

$W(t, \alpha)$ for large times

The asymptotic expression of T valid for $|\arg s| < \pi$ is

$$T(\tfrac{1}{2}s, \alpha) \sim \frac{1}{2}\, \frac{1 \pm i\alpha^2(1 - 2\hat{G})\, e^{-s}}{1 \pm \frac{1}{2}i\alpha^2 s^{-1} e^{-s}}. \tag{65}$$

By setting the denominator equal to zero, $T(\frac{1}{2}s, \alpha)$ is found to have infinitely many simple poles to the left of, and bounded away from, the imaginary $s-$ axis.

It is shown by Ring (1960) that the integration in equation (61) along C_3 and C_4 (figure 2) is $o(1)$ as $n \to \infty$ for $t > 1$. Therefore (61) can be written as

$$W(t,\alpha) = \frac{1}{2\pi i}\int_{C_2} \frac{e^{st}}{s} T(\tfrac{1}{2}s, \alpha)\, ds + \Sigma\, \text{Res}\left(T(\tfrac{1}{2}s,\alpha)\frac{e^{st}}{s}\right), \tag{66}$$

where the summation is to be made over all poles to the right of C_2. The summation can be shown to converge for $t > 1$, by using the asymptotic formulation at the poles.

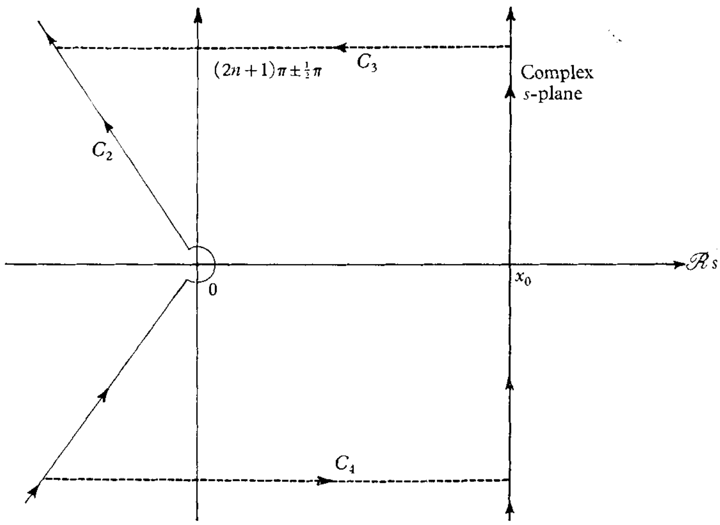


FIGURE 2. Integration contours in the evaluation of $W(t, \alpha)$.

It is now assumed that $T(\frac{1}{2}s, \alpha)$ has no poles for $\mathscr{R}s \geqslant 0$, i.e. that there are no unstable or neutrally stable solutions with $\Gamma_0^{(1)} \equiv 0$. By using the expansion of T for small s, the integral in equation (66) for large times can be given directly (e.g. Doetsch 1950). Hence,

$$W(t,\alpha) = (1-\alpha^2) - \tfrac{1}{2}(1+\alpha^2)\,t^{-1} + \dots. \tag{67}$$

The summation term in equation (66) does not contribute under the asumption that there are no poles for $\mathscr{R}s \geqslant 0$ and because the effect of any poles with $\mathscr{R}s < 0$ (which do in fact exist) is exponentially small. In this connexion, it should be recalled that these poles are bounded away from the imaginary axis.

Equation (67) is seen to reduce to the classical case for $\alpha^2 = 0$. As time goes to infinity, the solution approaches the Sears–Resler solution; the time to reach the steady solution is increased by the factor $(1+\alpha^2)$. This effect might be expected in view of the lagging of the magnetic field.

8. Flow with $\alpha^2 > 1$

General relations

The theory for general unsteady-airfoil motion has been formulated in §5 in terms of an arbitrary circulation $\Gamma_k^{(1)}$, where $\Gamma_k^{(1)}$ is zero for a trailing-edge Kutta condition. For $\alpha^2 > 1$, the Kutta condition is not directly applicable; also, the proper extension is not obvious, as noted in §4. Without a law to specify $\Gamma_k^{(1)}$, the flow cannot be determined uniquely. However, some general information regarding the stability of the flow can be obtained. For this purpose it is convenient to express $\Gamma_k^{(1)}$ in terms of the leading- and trailing-edge singularity strengths.

Let us introduce the ratio, β, of $\Gamma_k^{(1)}(t)$ to the other part of the circulation, i.e.

$$\Gamma_k^{(1)}(t) = -\beta\left\{2\int_0^1 \frac{DY(\xi,t)}{Dt}\frac{d\xi}{\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}}+\int_{\text{wake}} \epsilon^{(1)}(\xi,t)\left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}}\frac{d\xi}{\xi}\right\}. \tag{68}$$

It is easily shown that

$$\frac{\gamma^{(1)}(1^-,t)}{\gamma^{(1)}(0^+,t)} = \frac{\text{trailing-edge singularity strength}}{\text{leading-edge singularity strength}} = \frac{-\beta}{1-\beta}, \tag{69}$$

so that β is a measure of the relative singularity strengths. $\beta = 0$ corresponds to a trailing-edge Kutta condition and $\beta = 1$ corresponds to a leading-edge Kutta condition. It seems reasonable to restrict β to the range $O \leqslant \beta \leqslant 1$, since β outside this range corresponds to 'supercirculation.' This will be done here although it is not a critical assumption for the analysis.

Harmonic oscillation

Consider an oscillating airfoil with the upwash as given by equation (31). Then the quasi-steady quantities and wake distributions can again be represented as in equations (32) to (35). Using these, the analogous Wagner integral equations, (15) and (17), are treated in a manner similar to that leading to (36) and (38). The difference is that the integration over ϵ_1 is forward of the airfoil; this has the net effect of changing the sign of the K_0 terms as indicated in the definitions (37). These results can be reduced to

$$-2G^{(1)}-2\beta G_k^{(1)} = g_1[(1-2\beta)K_0^{(1)}+K_1^{(1)}]+g_2[(1-2\beta)K_0^{(2)}+K_1^{(2)}], \tag{70}$$

$$-2G^{(2)}-2\beta J_0 G_k^{(1)} = g_1 D_\beta(\omega,\alpha)+g_2 D_\beta(\omega,-\alpha), \tag{71}$$

where $\beta G_k^{(1)}$ corresponds to the quasi-steady contribution of equation (68),

$$G_k^{(1)} = -2\int_0^1 g(\xi)\frac{d\xi}{\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}}, \tag{72}$$

and the function D (see (40)) is modified to include the wake effects associated with β, giving

$$D_\beta(\omega,\alpha) = [(1-2\beta)J_0 - iJ_1]K_0^{(1)}+\alpha^{-1}[J_0K_1^{(1)}+iJ_1K_0^{(1)}]. \tag{73}$$

The lift and pitching moment can be given in a form similar to equations (43) and (45). This will not be carried out in detail here; rather, let us turn our attention to the stability of the flow.

Stability of the flow

For a given airfoil motion and a given β, the wake intensities g_1 and g_2 are given by the solution of equations (70) and (71), provided the determinant of the coefficients does not vanish. If the determinant vanishes, then there is an eigensolution, i.e. there is a solution with $g(x) \equiv 0$ but $g_{1,2} \neq 0$. Let us look for such solutions.

The determinant of the coefficients of g_1 and g_2 in equations (70) and (71) is

$$\begin{vmatrix} (1-2\beta)K_0^{(1)}+K_1^{(1)} & (1-2\beta)K_0^{(2)}+K_1^{(2)} \\ D_\beta(\omega,\alpha) & D_\beta(\omega,-\alpha) \end{vmatrix} \sim 4\left\{\frac{(1-\alpha^2)\pi}{i\omega^3}\right\}^{\frac{1}{2}} \exp\left\{\tfrac{1}{2}i\omega\left(1+\frac{2\alpha}{1-\alpha^2}\right)\right\}$$

$$\times\left\{-\beta\frac{\alpha+1}{\alpha}+(1-\beta)\frac{\alpha-1}{\alpha}\,i\,e^{-i\omega}+\frac{1}{4i\omega}\left(-2\beta\alpha-\frac{(2-\beta)(\alpha-1)}{\alpha}\right.\right.$$

$$\left.\left.+\left[2\alpha(1-\beta)-\frac{(1+\beta)(\alpha+1)}{\alpha}\right]i\,e^{-i\omega}\right)+\ldots\right\}, \tag{74}$$

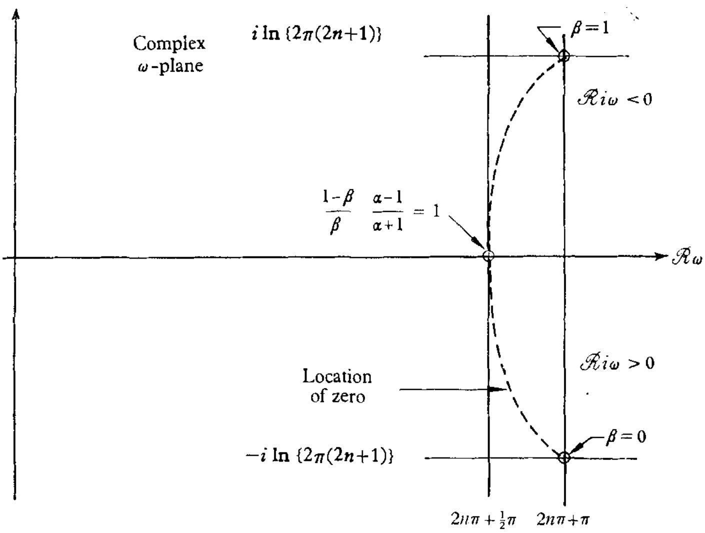


FIGURE 3. Location of an eigen-value, corresponding to a given positive integer n, in the complex ω-plane as a function of α and β. $\beta = 0$ corresponds to a trailing-edge Kutta condition; $\beta = 1$ to a Kutta condition applied at the leading-edge.

where the asymptotic expansion is valid for $|\arg\omega| < \pi$. The zeros of the determinant for large frequencies can be found by equating the largest terms in the expansion. If $0 < \beta < 1$, then we take

$$-\beta(\alpha+1)+(1-\beta)(\alpha-1)\,i\,e^{-i\omega} = 0$$

which gives zeros at

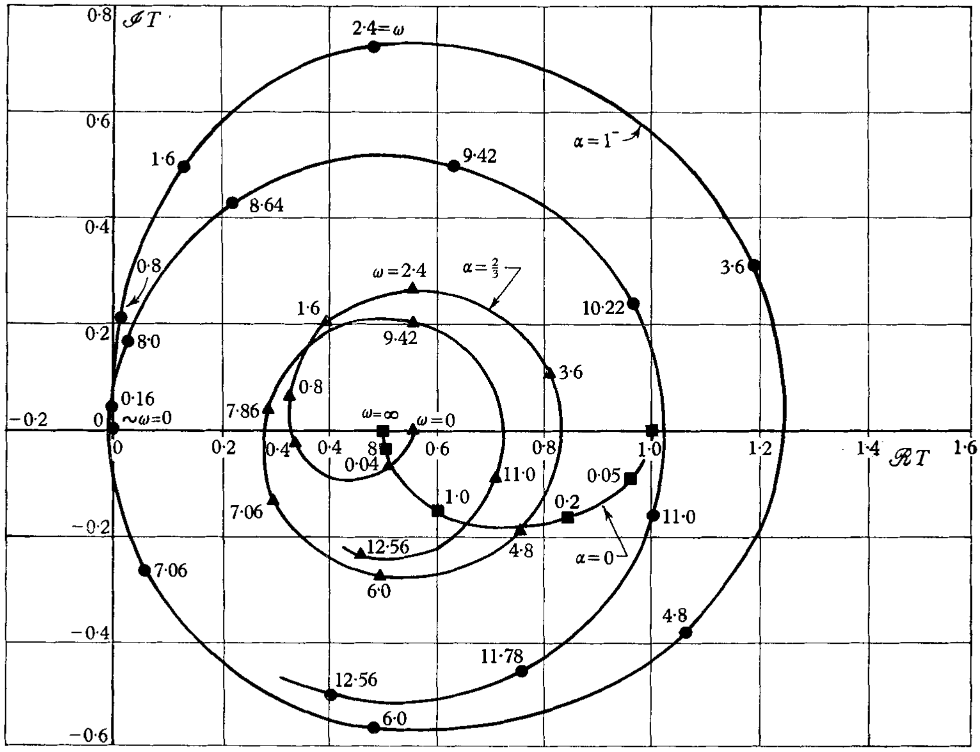
$$\mathscr{R}\omega = 2\pi n+\tfrac{1}{2}\pi, \quad \mathscr{R}i\omega = \ln\{(1-\beta)(\alpha-1)/\beta(\alpha+1)\}, \tag{75}$$

where n is a large positive integer. Continuing this analysis, it is found that a given pole will move as shown in figure 3, as β goes from 0 to 1. A similar analysis shows that there is another set of poles which corresponds to reflexion about the imaginary ω-axis in figure 3.

Therefore the determinant of equation (74) has infinitely many zeros, with corresponding eigen-solutions. For $\mathscr{R}i\omega > 0$, the solutions diverge exponentially. Thus, whenever

$$\beta(\alpha+1)/(1-\beta)(\alpha-1) < 1, \tag{76}$$

(a)



(b)

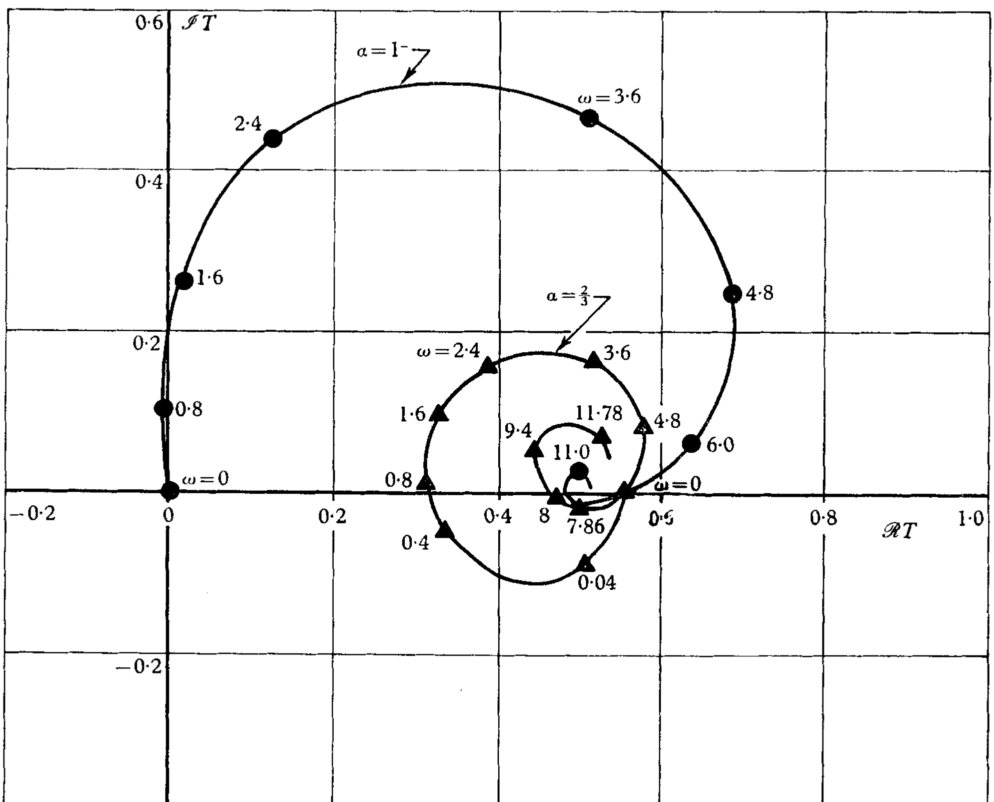


FIGURE 4(*a*). The modified Theodorsen function, $T(\frac{1}{2}i\omega, \alpha)$, for vertical oscillations, case I. This is typical of the general case. (*b*) The modified Theodorsen function, $T(\frac{1}{2}\omega, \alpha)$, for the case of a downwash distribution linear about the airfoil quarter-chord, case III. This is the special case in which the function is not asymptotic to a circle.

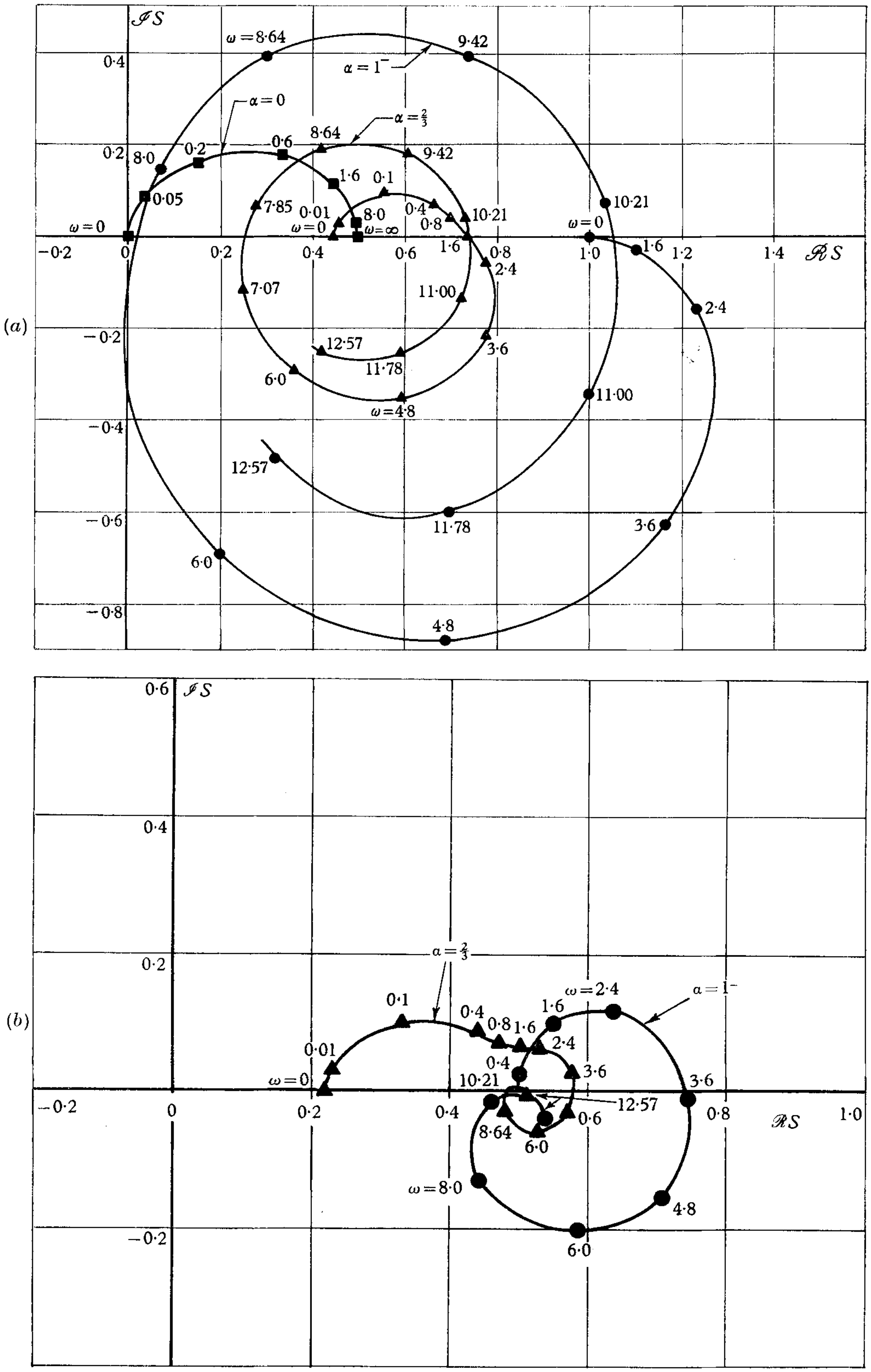


FIGURE 5(a). The function $S(\frac{1}{2}i\omega, \alpha)$ for vertical oscillation, case I. (b) The function $S(\frac{1}{2}i\omega, \alpha)$ for a downwash distribution linear about the airfoil quarter-chord, case III.

there will be diverging eigen-solutions and the flow will be unstable. Using (69), this criterion can be written as†

$$\left|\frac{\text{Trailing edge singularity strength}}{\text{Leading edge singularity strength}}\times\frac{\text{Rear wake speed}}{\text{Forward wake speed}}\right| < 1. \tag{77}$$

We conclude, from this brief study of flow with $\alpha^2 > 1$, that steady flow must be unstable when the criterion stated in (76) and (77) is satisfied. It appears that under these conditions the vortices that move forward and rearward are capable of interacting in such a way as to induce a diverging oscillatory circulation about the stationary airfoil. Presumably such oscillation would continue until violent separation (stalling) occurs, invalidating the assumptions of the present theory.

On the other hand, it is conceivable that the resultant effect of viscosity and electrical resistance in a real fluid, when $\alpha^2 > 1$, is to enforce an effective Kutta condition at the leading, rather than at the trailing, edge. This possibility is suggested by the studies of Lary (1960), Lewellen (1959), Greenspan & Carrier (1959), and Hasimoto (1959) (all of which pertain, however, to steady flow). If this occurs, criterion (76) may not be satisfied, and we cannot conclude here that the flow is unstable. It seems clear that further study in this area must await clarification of the roles of viscous and magnetic boundary layers and separation in fixing the circulation about cylindrical bodies.

9. Conclusions

The main conclusion to be drawn from this work is that unsteady effects become more important as the magnetic field strength increases. The steady-flow theories of Lary (1960) and of Sears & Resler (1959) predict small forces for α^2 near unity; the present theory predicts large forces due to unsteady effects. In fact, figures 4 and 5 show that at moderate frequencies the forces may be larger than in the corresponding non-magnetic case.

For $\alpha^2 > 1$, there is a definite possibility that the flow will be unstable, i.e., that no steady solution exists. In particular, if the boundary layers (magnetic and/or viscous) act in such a way as to remove the trailing-edge singularity, the flow will be unstable.

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† It is interesting to examine the relative wake intensities for the eigen-solutions. This has been carried out, using the asymptotic form of (70) together with (35). It is found that the vortex flux leaving the leading edge is larger than the vortex flux leaving the trailing edge in all the unstable eigen-solutions.

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